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A study on special curves of AW(k)-type in the pseudo-Galilean space

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Abstract. This paper is devoted to the study of AW(k)-type ($1 \leq k \leq 3$) curves according to the equiform differential geometry of the pseudo-Galilean space G_3^1. We show that equiform Bertrand curves are circular helices or isotropic circles of G_3^1. Also, there are equiform Bertrand curves of AW(3) and weak AW(3)-types. Moreover, we give the relations between the equiform curvatures of these curves. Finally, examples of some special curves are given and plotted.

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1 Introduction

As it is well known, geometry of space is associated with mathematical group. The idea of invariance of geometry under transformation group may imply that, on some spacetimes of maximum symmetry there should be a principle of relativity which requires the invariance of physical laws without gravity under transformations among inertial systems [1]. Besides, theory of curves and the curves of constant curvature in the equiform differential geometry of the isotropic spaces I_3^1 , I_3^2 and the Galilean space G_3 are described in [2] and [3], respectively. The pseudo-Galilean space is one of the real Cayley-Klein spaces. It has projective signature $(0, 0, +, -)$ according to [2]. The absolute of the pseudo-Galilean space is an ordered triple $\{w, f, I\}$ where w is the ideal plane, f a line in w and I is the fixed hyperbolic involution of the points of f. In [4], from the differential geometric point of view, K. Arslan and A. West defined the notion of AW(k)-type submanifolds. Since then, many works have been done related to AW(k)-type submanifolds (see, for example, [5–10]). In [9], Özgür and Gezgin studied a Bertrand curve of AW(k)-type and furthermore, they showed that there is no such Bertrand curve of AW(1) and AW(3)-types if and only if it is a right circular helix. In addition, they studied weak AW(2)-type and AW(3)-type conical geodesic curves in Euclidean 3-space E^3.

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Besides, In 3-dimensional Galilean space and Lorentz space, the curves of AW(k)-type were investigated in [6,8]. In [7], the authors gave curvature conditions and characterizations related to AW(k)-type curves in E^n and in [10], the authors investigated curves of AW(k)-type in the 3-dimensional null cone.

In this paper, to the best of author's knowledge, Bertrand curves of AW(k)-type have not been presented in the equiform geometry of the pseudo-Galilean space G_3^1 in depth. Thus, the study is proposed to serve such a need. Our paper is organized as follows. In Section 2, the basic notions and properties of a pseudo-Galilean geometry are reviewed. In Section 3, properties of the equiform geometry of the pseudo-Galilean space G_3^1 are given. Section 4 contains a study of AW(k)-type equiform Frenet curves. Equiform Bertrand curves of AW(k)-type in G_3^1 included in section 5.

2 Pseudo-Galilean geometric meanings

In this section, let us first recall basic notions from pseudo-Galilean geometry [11,12]. In the inhomogeneous affine coordinates for points and vectors (point pairs) the similarity group H_8 of G_3^1 has the following form

$$\begin{aligned}
\bar{x} &= a + b.x, \\
\bar{y} &= c + d.x + r.\cosh\theta.y + r.\sinh\theta.z, \\
\bar{z} &= e + f.x + r.\sinh\theta.y + r.\cosh\theta.z,
\end{aligned} \tag{2.1}$$

where a, b, c, d, e, f, r and θ are real numbers. Particularly, for $b = r = 1$, the group (2.1) becomes the group $B_6 \subset H_8$ of isometries (proper motions) of the pseudo-Galilean space G_3^1. The motion group leaves invariant the absolute figure and defines the other invariants of this geometry. It has the following form

$$\begin{aligned}
\bar{x} &= a + x, \\
\bar{y} &= c + d.x + \cosh\theta.y + \sinh\theta.z, \\
\bar{z} &= e + f.x + \sinh\theta.y + \cosh\theta.z.
\end{aligned} \tag{2.2}$$

According to the motion group in the pseudo-Galilean space, there are non-isotropic vectors $A(A_1, A_2, A_3)$ (for which holds $A_1 \neq 0$) and four types of isotropic vectors: spacelike ($A_1 = 0$, $A_2^2 - A_3^2 > 0$), timelike ($A_1 = 0$, $A_2^2 - A_3^2 < 0$) and two types of lightlike vectors ($A_1 = 0, A_2 = \pm A_3$). The scalar product of two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ in G_3^1 is defined by

$$\langle u, v\rangle = \left\{ \begin{array}{ll} u_1 v_1, & \text{if } u_1 \neq 0 \text{ or } v_1 \neq 0, \\ u_2 v_2 - u_3 v_3 & \text{if } u_1 = 0 \text{ and } v_1 = 0. \end{array} \right\}$$

We introduce a pseudo-Galilean cross product in the following way

$$u \times_{G_3^1} v = \begin{vmatrix} 0 & -j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix},$$

where $j = (0,1,0)$ and $k = (0,0,1)$ are unit spacelike and timelike vectors, respectively. Let us recall basic facts about curves in G_3^1, that were introduced in [15].

A curve $\gamma(s) = (x(s), y(s), z(s))$ is called an admissible curve if it has no inflection points ($\dot{\gamma} \times \ddot{\gamma} \neq 0$) and no isotropic tangents ($\dot{x} \neq 0$) or normals whose projections on the absolute plane would be lightlike vectors ($\dot{y} \neq \pm \dot{z}$). An admissible curve in G_3^1 is an analogue of a regular curve in Euclidean space [12].

For an admissible curve $\gamma : I \subseteq \mathbb{R} \to G_3^1$, the curvature $\kappa(s)$ and torsion $\tau(s)$ are defined by

$$\kappa(s) = \frac{\sqrt{|\ddot{y}(s)^2 - \ddot{z}(s)^2|}}{(\dot{x}(s))^2}, \ \tau(s) = \frac{\ddot{y}(s)\dddot{z}(s) - \dddot{y}(s)\ddot{z}(s)}{|\dot{x}(s)|^5 \cdot \kappa^2(s)}, \tag{2.3}$$

expressed in components. Hence, for an admissible curve $\gamma : I \subseteq \mathbb{R} \to G_3^1$ parameterized by the arc length s with differential form $ds = dx$, given by

$$\gamma(x) = (x, y(x), z(x)), \tag{2.4}$$

the formulas (2.3) have the following form

$$\kappa(x) = \sqrt{|y''(x)^2 - z''(x)^2|}, \ \tau(x) = \frac{y''(x)z'''(x) - y'''(x)z''(x)}{\kappa^2(x)}. \tag{2.5}$$

The associated trihedron is given by

$$\begin{aligned} \mathbf{e}_1 &= \gamma'(x) = (1, y'(x), z'(x)), \\ \mathbf{e}_2 &= \frac{1}{\kappa(x)}\gamma''(x) = \frac{1}{\kappa(x)}(0, y''(x), z''(x)), \\ \mathbf{e}_3 &= \frac{1}{\kappa(x)}(0, \epsilon z''(x), \epsilon y''(x)), \end{aligned} \tag{2.6}$$

where $\epsilon = +1$ or $\epsilon = -1$, chosen by criterion $\det(e_1, e_2, e_3) = 1$, that means

$$\left| y''(x)^2 - z''(x)^2 \right| = \epsilon(y''(x)^2 - z''(x)^2).$$

The curve γ given by (2.4) is timelike (resp. spacelike) if $\mathbf{e}_2(s)$ is a spacelike (resp. timelike) vector. The principal normal vector or simply normal is spacelike if $\epsilon = +1$ and timelike if $\epsilon = -1$. For derivatives of the tangent $\mathbf{e}_1$, normal $\mathbf{e}_2$ and binormal $\mathbf{e}_3$ vector fields, the following Frenet formulas in G_3^1 hold:

$$\begin{aligned} \mathbf{e}_1'(x) &= \kappa(x)\mathbf{e}_2(x), \\ \mathbf{e}_2'(x) &= \tau(x)\mathbf{e}_3(x), \\ \mathbf{e}_3'(x) &= \tau(x)\mathbf{e}_2(x). \end{aligned} \tag{2.7}$$

3 Frenet formulas according to the equiform geometry of G_3^1

This section contains some important facts about equiform geometry. The equiform differential geometry of curves in the pseudo-Galilean space G_3^1 has been described in [11]. In the equiform geometry a few specific terms will be introduced. So, let $\gamma(s) : I \to G_3^1$ be an admissible curve in the pseudo-Galilean space G_3^1, the equiform parameter of γ is defined by

$$\sigma := \int \frac{1}{\rho} ds = \int \kappa ds,$$

where $\rho = \frac{1}{\kappa}$ is the radius of curvature of the curve γ. Then, we have

$$\frac{ds}{d\sigma} = \rho. \tag{3.1}$$

Let h be a homothety with the center in the origin and the coefficient μ. If we put $\bar{\gamma} = h(\gamma)$, then it follows

$$\bar{s} = \mu s \ \text{ and } \ \bar{\rho} = \mu\rho,$$

where $\bar{s}$ is the arc-length parameter of $\bar{\gamma}$ and $\bar{\rho}$ the radius of curvature of this curve. Therefore, σ is an equiform invariant parameter of γ [11].

Notation 3.1 *The functions κ and τ are not invariants of the homothety group, then from* (2.3) *it follows that $\bar{\kappa} = \frac{1}{\mu}\kappa$ and $\bar{\tau} = \frac{1}{\mu}\tau$.*

From now on, we define the Frenet formulas of the curve γ with respect to its equiform invariant parameter σ in G_3^1. The vector

$$\mathbf{T} = \frac{d\gamma}{d\sigma},$$

is called a tangent vector of the curve γ. From (2.6) and (3.1) we get

$$\mathbf{T} = \frac{d\gamma}{ds}\frac{ds}{d\sigma} = \rho \cdot \frac{d\gamma}{ds} = \rho \cdot \mathbf{e}_1. \tag{3.2}$$

Further, we define the principal normal vector and the binormal vector by

$$\mathbf{N} = \rho \cdot \mathbf{e}_2, \quad \mathbf{B} = \rho \cdot \mathbf{e}_3. \tag{3.3}$$

It is easy to show that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is an equiform invariant frame of γ. On the other hand, the derivatives of these vectors with respect to σ are given by

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}' = \begin{bmatrix} \dot{\rho} & 1 & 0 \\ 0 & \dot{\rho} & \rho\tau \\ 0 & \rho\tau & \dot{\rho} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{3.4}$$

The functions $\mathcal{K} : I \to \mathbb{R}$ defined by $\mathcal{K} = \dot{\rho}$ is called the equiform curvature of the curve γ and $\mathcal{T} : I \to \mathbb{R}$ defined by $\mathcal{T} = \rho\tau = \frac{\tau}{\kappa}$ is called the equiform torsion of this curve. In the light

of this, the formulas (3.4) analogous to the Frenet formulas in the equiform geometry of the pseudo-Galilean space G_3^1 can be written as

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}' = \begin{bmatrix} \mathcal{K} & 1 & 0 \\ 0 & \mathcal{K} & \mathcal{T} \\ 0 & \mathcal{T} & \mathcal{K} \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{3.5}$$

The equiform parameter $\sigma = \int \kappa(s)ds$ for closed curves is called the total curvature, and it plays an important role in global differential geometry of Euclidean space. Also, the function $\frac{\tau}{\kappa}$ has been already known as a conical curvature and it also has interesting geometric interpretation.

Notation 3.2 *Let $\gamma : I \to G_3^1$ be a Frenet curve in the equiform geometry of the G_3^1, the following statements are true (see for details [11, 13]):*

1. If $\gamma(s)$ is an isotropic logarithmic spiral in G_3^1. Then, $\mathcal{K}$ =const. $\neq 0$ and $\mathcal{T} = 0$,
2. If $\gamma(s)$ is a circular helix in G_3^1. Then, $\mathcal{K}$ =0 and $\mathcal{T}$ =const. $\neq 0$,
3. If $\gamma(s)$ is an isotropic circle in G_3^1. Then, $\mathcal{K}$ =0 and $\mathcal{T} = 0$.

4 AW(k)-type curves in the equiform geometry of G_3^1

Let $\gamma : I \to G_3^1$ be a curve in the equiform geometry of the pseudo-Galilean space G_3^1. The curve γ is called a Frenet curve of osculating order l if its derivatives $\gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(l)}(s)$ are linearly dependent and $\gamma'(s), \gamma''(s), \gamma'''(s), ..., \gamma^{(l+1)}(s)$ are no longer linearly independent for all $s \in I$. To each Frenet curve of order 3 one can associate an orthonormal 3-frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ along γ, such that $\gamma'(s) = \frac{1}{\rho}\mathbf{T}$, called the equiform Frenet frame (Eqs. (3.5)).

Now, we consider equiform Frenet curevs of osculating order 3 in G_3^1 and start with some important results.

Let $\gamma : I \to G_3^1$ be a Frenet curve in the equiform geometry of the pseudo-Galilean space. By the use of Frenet formulas (3.5), we obtain the higher order derivatives of γ as follows

$$\begin{aligned}
\gamma'(s) &= \frac{d\gamma}{d\sigma}\frac{d\sigma}{ds} = \frac{1}{\rho}\mathbf{T}, \\
\gamma''(s) &= \frac{1}{\rho^2}\mathbf{N}, \\
\gamma'''(s) &= \frac{1}{\rho^3}\left(-\mathcal{K}\mathbf{N}+\mathcal{T}\mathbf{B}\right), \\
\gamma''''(s) &= \frac{1}{\rho^4}[(2\mathcal{K}^2+\mathcal{T}^2 - \mathcal{K}')\mathbf{N} + (\mathcal{T}' - 3\mathcal{K}\mathcal{T})\mathbf{B}].
\end{aligned}$$

Notation 4.1 *Let us write*

$$Q_1 = \frac{1}{\rho^2}\mathbf{N}, \tag{4.1}$$

$$Q_2 = \frac{1}{\rho^3}\left(-\mathcal{K}\mathbf{N}+\mathcal{T}\mathbf{B}\right), \tag{4.2}$$

$$Q_3 = \frac{1}{\rho^4}[(2\mathcal{K}^2+\mathcal{T}^2-\mathcal{K}')\mathbf{N}+(\mathcal{T}'-3\mathcal{K}\mathcal{T})\mathbf{B}]. \tag{4.3}$$

Notation 4.2 *$\gamma'(s), \gamma''(s), \gamma'''(s)$ and $\gamma''''(s)$ are linearly dependent if and only if Q_1, Q_2 and Q_3 are linearly dependent.*

Definition 4.1 *Frenet curves (of osculating order 3) in the equiform geometry of the pseudo-Galilean space G_3^1 are called [5]:*

1. *of type equiform $AW(1)$ if they satisfy $Q_3 = 0$,*
2. *of type equiform $AW(2)$ if they satisfy $\|Q_2\|^2 \, Q_3 = \langle Q_3(s), Q_2\rangle Q_2$,*
3. *of type equiform $AW(3)$ if they satisfy $\|Q_1\|^2 \, Q_3 = \langle Q_3, Q_1(s)\rangle Q_1$,*
4. *of type weak equiform $AW(2)$ if they satisfy*

$$Q_3 = \langle Q_3, Q_2^* \rangle \, Q_2^*, \tag{4.4}$$

5. *of type weak equiform $AW(3)$ if they satisfy*

$$Q_3 = \langle Q_3, Q_1^* \rangle \, Q_1^*, \tag{4.5}$$

where

$$\begin{aligned} Q_1^* &= \frac{Q_1}{\|Q_1\|}, \\ Q_2^* &= \frac{Q_2 - \langle Q_2, Q_1^* \rangle \, Q_1^*}{\|Q_2 - \langle Q_2, Q_1^* \rangle \, Q_1^*\|}. \end{aligned} \tag{4.6}$$

Proposition 4.1 *Let $\gamma : I \to G_3^1$ be a Frenet curve (of osculating order 3) in the equiform geometry of the pseudo-Galilean space G_3^1,*

(i) γ is of type weak equiform $AW(2)$ if and only if

$$2\mathcal{K}^2 + \mathcal{T}^2 - \mathcal{K}' = 0, \tag{4.7}$$

(ii) γ is of type weak equiform $AW(2)$ if and only if

$$\mathcal{T}' - 3\mathcal{K}\mathcal{T}(s) = 0. \tag{4.8}$$

Proof. According to Definition 4.1 and Notation 4.1, the proof is obvious. ■

Theorem 4.1 *Let $\gamma : I \to G_3^1$ be a Frenet curve (of osculating order 3) in the equiform geometry of the pseudo-Galilean space G_3^1. Then γ is of type equiform AW(2) if and only if*

$$-\mathcal{K}' + 2\mathcal{K}^2 + \mathcal{T}^2 = 0,$$

$$3\mathcal{K}\mathcal{T} - \mathcal{T}' = 0. \tag{4.9}$$

Proof. Since γ is of type equiform AW(2), then from (4.3), we obtain

$$\frac{1}{\rho^4}[(2\mathcal{K}^2+\mathcal{T}^2(s) - \mathcal{K}')\mathbf{N} + (\mathcal{T}' - 3\mathcal{K}\mathcal{T})\mathbf{B}] = 0.$$

As we know, the vectors $\mathbf{N}$ and $\mathbf{B}$ are linearly independent, so we can write

$$2\mathcal{K}^2+\mathcal{T}^2 - \mathcal{K}' = 0 \text{ and } \mathcal{T}' - 3\mathcal{K}\mathcal{T} = 0.$$

The converse statement is straightforward and therefore the proof is completed. ■

Theorem 4.2 *Let $\gamma : I \to G_3^1$ be a Frenet curve (of osculating order 3) in the equiform geometry of the pseudo-Galilean space G_3^1. Then γ is of type equiform AW(2) if and only if*

$$\mathcal{K}^2\mathcal{T} - \mathcal{K}\mathcal{T}' + \mathcal{T}\mathcal{K}' - \mathcal{T}^3 = 0. \tag{4.10}$$

Proof. Assuming that γ is a Frenet curve in the equiform geometry of G_3^1 , then from (4.2) and (4.3), one can write

$$Q_2 = a_{11}\mathbf{N} + a_{12}\mathbf{B},$$
$$Q_3 = a_{21}\mathbf{N} + a_{22}\mathbf{B},$$

where a_{11},a_{12}, a_{21} and a_{22} are differentiable functions. Since Q_2 and Q_3 are linearly dependent, coefficients determinant equals zero and hence

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0, \tag{4.11}$$

where

$$\begin{aligned} a_{11} &= \frac{-1}{\rho^3}\mathcal{K},\ a_{12} = \frac{1}{\rho^3}\mathcal{T}, \\ a_{21} &= \frac{1}{\rho^4}[-\mathcal{K}' + 2\mathcal{K}^2 + \mathcal{T}^2], \\ a_{22} &= \frac{1}{\rho^4}[-3\mathcal{K}\mathcal{T} + \mathcal{T}']. \end{aligned} \tag{4.12}$$

From (4.11) and (4.12), we obtain (4.10). It can be easily shown that the converse assertion is also true. ■

Corollary 4.1 *Let $\gamma : I \to G_3^1$ be a Frenet curve (of osculating order3) in the equiform geometry of the pseudo-Galilean space G_3^1,*

(i) If γ is an isotropic logarithmic spiral in G_3^1, then γ is of equiform $AW(2)$-type curve.

(ii) If γ is an equiform space or timelike general (circular) helix in G_3^1, then it is not of equiform $AW(k)$, weak $AW(2)$ and weak $AW(3)$-types.

Theorem 4.3 *Let $\gamma : I \to G_3^1$ be a Frenet curve (of osculating order 3) in the equiform geometry of G_3^1. Then γ is of equiform $AW(3)$-type if and only if*

$$\mathcal{T}' - 3\mathcal{K}\mathcal{T} = 0. \tag{4.13}$$

Proof. Using Definition 4.1 and Eqs. (4.1) and (4.3), we obtain (4.13). The converse direction is obvious, hence our Theorem is proved. ■

5 Bertrand curves of AW(k)-type

Definition 5.1 *A curve $\gamma : I \to G_3^1$ with equiform curvature $\mathcal{K} = 0$ is called an equiform Bertrand curve if there exist a curve $\bar{\gamma} : I \to G_3^1$ with equiform curvature $\bar{\mathcal{K}} = 0$ such that the principal normal lines of γ and $\bar{\gamma}$ are parallel at the corresponding points. In this case $\bar{\gamma}$ is called an equifrm Bertrand mate of γ and vise versa.*

By Definition 5.1, we can say that for given an equiform Bertrand pair $(\gamma, \bar{\gamma})$, there exist a functional relation $\bar{s} = \bar{s}(s)$ such that $\lambda(\bar{s}(s) = \lambda(s)$, then the equiform Bertrand mate of γ is given by

$$\bar{\gamma}(s) = \gamma(s) + \lambda\mathbf{N}. \tag{5.1}$$

Theorem 5.1 *If $(\gamma, \bar{\gamma})$ is an equiform Bertrand pair in the equiform geometry of the pseudo-Galilean space G_3^1, then*

(i) *The function λ is constant.*

(ii) *γ with non-zero constant equiform torsion is a circular helix in G_3^1.*

(iii) *γ with zero equiform torsion is an isotropic circle of G_3^1*

Proof. Along γ and $\bar{\gamma}$, let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ and $\{\bar{\mathbf{T}}, \bar{\mathbf{N}}, \bar{\mathbf{B}}\}$ be the Frenet frames according to the equiform geometry of the pseudo-Galilean space G_3^1, respectively. Differentiate (5.1) with respect to s, we obtain

$$\bar{\mathbf{T}} = \mathbf{T} + \lambda\mathbf{N}' + \lambda'\mathbf{N}. \tag{5.2}$$

By using (3.5), we have

$$\bar{\mathbf{T}} = \mathbf{T} + \left(\lambda\mathcal{K} + \lambda'\right)\mathbf{N} + \lambda\mathcal{T}\mathbf{B}.$$

Since $\bar{\mathbf{N}}$ is parallel to $\mathbf{N}$, we get

$$\lambda\mathcal{K} + \lambda' {=} 0,$$

it follows that

$$\lambda = const.$$

If γ has a non-zero constant equiform torsion, then γ is characterized by

$$\kappa = const. \neq 0,\ \ \tau = const. \neq 0,$$

and therefore $\tau/\kappa = const.$holds.

On the other hand, whenever $\mathcal{T} = 0$, the natural equations of γ is given by

$$\kappa = const. \neq 0,\ \ \tau = 0,$$

and so, the curve γ is an isotropic circle in G_3^1 [14]. Thus the proof is completed. ■

Theorem 5.2 *If $(\gamma, \bar{\gamma})$ is a Bertrand pair in the equiform geometry of the pseudo-Galilean space G_3^1, then the angle between tangent vectors at corresponding points is constant.*

Proof. To prove that the angle is constant, we need to show that $\left\langle \bar{\mathbf{T}}, \mathbf{T}\right\rangle' = 0$. For this purpose using (3.5) to obtain

$$\begin{aligned}\left\langle \bar{\mathbf{T}}, \mathbf{T}\right\rangle' &= \left\langle \bar{\mathbf{T}}', \mathbf{T}\right\rangle + \left\langle \bar{\mathbf{T}}, \mathbf{T}'\right\rangle \\ &= \left\langle \bar{\mathcal{K}}\bar{\mathbf{T}} + \bar{\mathbf{N}}, \mathbf{T}\right\rangle + \left\langle \bar{\mathbf{T}}, \mathcal{K}\mathbf{T} + \mathbf{N}\right\rangle \\ &= \tilde{\mathcal{K}}\left\langle \bar{\mathbf{T}}, \mathbf{T}\right\rangle + \left\langle \bar{\mathbf{N}}, \mathbf{T}\right\rangle + \mathcal{K}\left\langle \bar{\mathbf{T}}, \mathbf{T}\right\rangle \\ &\quad + \left\langle \bar{\mathbf{T}}, \mathbf{N}\right\rangle. \end{aligned} \tag{5.3}$$

Because of $\bar{\mathbf{N}}$ is parallel to $\mathbf{N}$, then

$$\left\langle \bar{\mathbf{N}}, \mathbf{T}\right\rangle = 0, \left\langle \bar{\mathbf{T}}, \mathbf{N}\right\rangle = 0. \tag{5.4}$$

Since $(\gamma, \bar{\gamma})$ is a Berrand pair in the equiform geometry of G_3^1, then from Theorem 5.1, we have

$$\mathcal{K} {=} 0 \text{ and } \bar{\mathcal{K}} {=} 0. \tag{5.5}$$

After substituting (5.4) and (5.5) into (5.3), we get

$$\left\langle \bar{\mathbf{T}}, \mathbf{T}\right\rangle' = 0. \tag{5.6}$$

In the light of (5.6) the angle between $\bar{\mathbf{T}}, \mathbf{T}$ is constant. Thus this completes the proof. ■

Corollary 5.1 *Let $\gamma(s) : I \to G_3^1$ be a Bertrand curve in the equiform geometry of G_3^1. Then*

(i) γ is a weak equiform $AW(3)$-type but not a weak equiform $AW(2)$-type.

(ii) γ is equiform $AW(3)$-type but not equiform $AW(1)$ and $AW(2)$-types.

6 Examples

We consider some examples (timelike and spacelike curves [11,12]) which characterize equiform general (circular) helices with respect to the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ in the equiform geometry of G_3^1 which satisfy some conditions of equiform curvatures ($\mathcal{K} = \mathcal{K}(s), \mathcal{T} = \mathcal{T}(s)$; $\mathcal{K}$ =*const.* $\neq 0, \mathcal{T}$ =*const.* $\neq 0$; $\mathcal{K}$ =*const.* $\neq 0, \mathcal{T}$ =0).

Example 6.1 *Consider the equiform **timelike** general helix $\mathbf{r} : I \longrightarrow G_3^1, I \subseteq \mathbb{R}$ parameterized by the arc length s with differential form $ds = dx$, given by*

$$\mathbf{r}(x) = (x, y(x), z(x)),$$

where

$$\begin{aligned} x(s) &= s, \\ y(s) &= \frac{e^{-as}}{(a^2-b^2)^2}\left(\left(a^2+b^2\right)\cosh(bs) + 2ab\sinh(bs)\right), \\ z(s) &= \frac{e^{-as}}{(a^2-b^2)^2}\left(2ab\cosh(bs) + \left(a^2+b^2\right)\sinh(bs)\right); \\ a, b &\in \mathbb{R} - \{0\}. \end{aligned}$$

The corresponding derivatives of $\mathbf{r}$ *are as follows*

$$\begin{aligned} \mathbf{r}' &= \left(1, \frac{-e^{-as}}{(a^2-b^2)}\left(a\cosh(bs) + b\sinh(bs)\right), \frac{e^{-as}}{(b^2-a^2)}\left(b\cosh(bs) + a\sinh(bs)\right)\right), \\ \mathbf{r}'' &= \left(0, e^{-as}\cosh(bs), e^{-as}\sinh(bs)\right), \\ \mathbf{r}''' &= \left(0, e^{-as}\left(-a\cosh(bs) + b\sinh(bs)\right), e^{-as}\left(b\cosh(bs) - a\sinh(bs)\right)\right). \end{aligned}$$

First of all, we find that the tangent vector of $\mathbf{r}$ *has the form*

$$\begin{aligned} \mathbf{e}_1 &= \left(x', y', z'\right) \\ &= \left(1, \frac{-e^{-as}}{(a^2-b^2)}\left(a\cosh(bs) + b\sinh(bs)\right), \frac{e^{-as}}{(b^2-a^2)}\left(b\cosh(bs) + a\sinh(bs)\right)\right). \end{aligned}$$

Then the two normals (normal and binormal) of the curve are, respectively

$$\begin{aligned} \mathbf{e}_2 &= \left(0, \cosh(bs), \sinh(bs)\right), \\ \mathbf{e}_3 &= \left(0, \sinh(bs), \cosh(bs)\right); \quad \det[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = 1. \end{aligned}$$

Thus the computations of the coordinate functions of $\mathbf{r}$ *lead to*

$$\kappa = e^{-as}, \ \tau = b \ .$$

From the equiform Frenet formulas (3.5) we can express vector fields $\mathbf{T}, \mathbf{N}, \mathbf{B}$ *as follows*

$$\begin{aligned}
\mathbf{T} &= \left(e^{as}, \frac{-1}{(a^2-b^2)}\left(a\cosh(bs)+b\sinh(bs)\right), \frac{1}{(b^2-a^2)}\left(b\cosh(bs)+a\sinh(bs)\right)\right), \\
\mathbf{N} &= \left(0, e^{as}\cosh(bs), e^{as}\sinh(bs)\right), \\
\mathbf{B} &= \left(0, e^{as}\sinh(bs), e^{as}\cosh(bs)\right),
\end{aligned}$$

respectively. In the light of this, the equiform curvatures are given by

$$\mathcal{K} = ae^{as}, \mathcal{T} = -be^{as}.$$

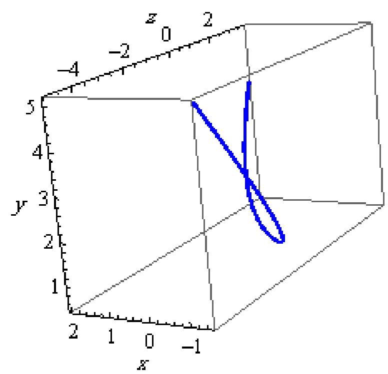

Figure 1: Equiform timelike general helix with $\mathcal{K}(s) = e^s, \mathcal{T}(s) = 2e^s$.

Example 6.2 *Let* $\mathbf{r} : I \longrightarrow G_3^1, I \subseteq \mathbb{R}$ *be the equiform* ***spacelike*** *general helix, given by*

$$\mathbf{r}(x) = (x, y(x), z(x)),$$

where

$$\begin{aligned}
x(s) &= s, \\
y(s) &= \frac{e^{-as}}{(a^2-b^2)^2}\left(2ab\cosh(bs)+\left(a^2+b^2\right)\sinh(bs)\right), \\
z(s) &= \frac{e^{-as}}{(a^2-b^2)^2}\left(\left(a^2+b^2\right)\cosh(bs)+2ab\sinh(bs)\right); \\
a, b &\in \mathbb{R}-\{0\}.
\end{aligned}$$

For the coordinate functions of $\mathbf{r}$*, we have*

$$\begin{aligned}
\mathbf{r}' &= \left(1, \frac{e^{-as}}{(b^2-a^2)}\left(b\cosh(bs)+a\sinh(bs)\right), \frac{-e^{-as}}{(a^2-b^2)}\left(a\cosh(bs)+b\sinh(bs)\right)\right), \\
\mathbf{r}'' &= \left(0, e^{-as}\sinh(bs), e^{-as}\cosh(bs)\right), \\
\mathbf{r}''' &= \left(0, e^{-as}\left(b\cosh(bs)-a\sinh(bs)\right), e^{-as}\left(b\sinh(bs)-a\cosh(bs)\right)\right).
\end{aligned}$$

Also, the associated trihedron is given by

$$\begin{aligned}
\mathbf{e}_1 &= \left(1, \frac{e^{-as}}{(b^2-a^2)}\left(b\cosh(bs)+a\sinh(bs)\right), \frac{-e^{-as}}{(a^2-b^2)}\left(a\cosh(bs)+b\sinh(bs)\right)\right),\\
\mathbf{e}_2 &= \left(0, \sinh(bs), \cosh(bs)\right),\\
\mathbf{e}_3 &= \left(0, -\cosh(bs), -\sinh(bs)\right).
\end{aligned}$$

The curvature and torsion of this curve are

$$\kappa = e^{-as},\ \tau = -b\ .$$

Furthermore, the tangent, normal and binormal vector fields in the equiform geometry of G_3^1 are obtained as follows

$$\begin{aligned}
\mathbf{T} &= \left(e^{as}, \frac{1}{(b^2-a^2)}\left(b\cosh(bs)+a\sinh(bs)\right), \frac{-1}{(a^2-b^2)}\left(a\cosh(bs)+b\sinh(bs)\right)\right),\\
\mathbf{N} &= \left(0, e^{as}\sinh(bs), e^{as}\cosh(bs)\right),\\
\mathbf{B} &= \left(0, -e^{as}\cosh(bs), -e^{as}\sinh(bs)\right),
\end{aligned}$$

respectively.

The equiform curvatures of $\mathbf{r}$ *are*

$$\mathcal{K} = ae^{as}, \mathcal{T} = -be^{as}.$$

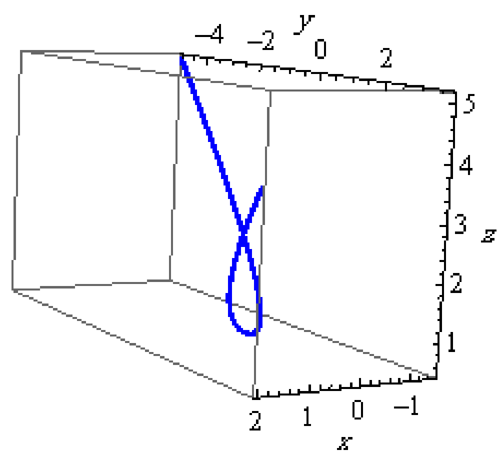

Figure 2: Equiform spacelike general helix with $\mathcal{K}(s) = e^s, \mathcal{T}(s) = -2e^s$.

Example 6.3 *In this example, let us consider the equiform timelike* ***circular*** *helix* $\mathbf{r} : I \longrightarrow G_3^1$ *given by*

$$\mathbf{r}(x) = (x, y(x), z(x)),$$

where

$$\begin{aligned} x(s) &= s, \\ y(s) &= \frac{a^3 s}{b\left(b^2 - a^2\right)} \left(b \sinh\left(\frac{b}{a}\ln(as)\right) - a\cosh\left(\frac{b}{a}\ln(as)\right)\right), \\ z(s) &= \frac{a^3 s}{b\left(b^2 - a^2\right)} \left(b \cosh\left(\frac{b}{a}\ln(as)\right) - a\sinh\left(\frac{b}{a}\ln(as)\right)\right); \\ a, b &\in \mathbb{R} - \{0\}. \end{aligned}$$

For this curve, the equiform vector fields are obtained as follows

$$\begin{aligned} \mathbf{T} &= \left(\frac{s}{a}, \frac{as}{b}\cosh\left(\frac{b}{a}\ln(as)\right), \frac{as}{b}\sinh\left(\frac{b}{a}\ln(as)\right)\right), \\ \mathbf{N} &= \left(0, \frac{s}{a}\sinh\left(\frac{b}{a}\ln(as)\right), \frac{s}{a}\cosh\left(\frac{b}{a}\ln(as)\right)\right), \\ \mathbf{B} &= \left(0, \frac{s}{a}\cosh\left(\frac{b}{a}\ln(as)\right), \frac{s}{a}\sinh\left(\frac{b}{a}\ln(as)\right)\right), \end{aligned}$$

respectively.

It follows that

$$\mathcal{K} = \frac{1}{a}, \mathcal{T} = \frac{-b}{a^2}.$$

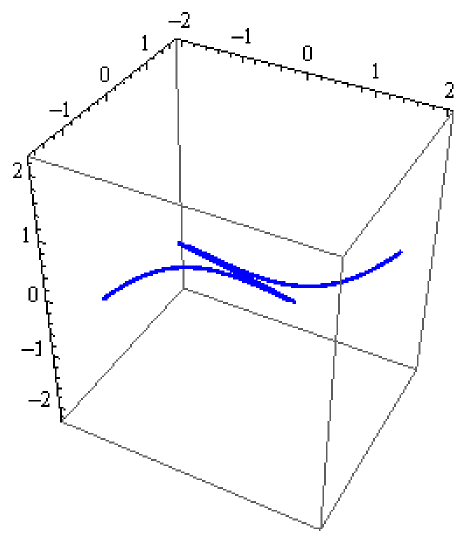

Figure 3: Equiform timelike circular helix with $\mathcal{K} = \frac{1}{a}, \mathcal{T} = \frac{-b}{a^2}$.

Example 6.4 *Let the equiform* ***spacelike*** *circular helix* $\mathbf{r}: I \longrightarrow G_3^1, I \subseteq \mathbb{R}$ *in the form*

$$\mathbf{r}(x) = (x, y(x), z(x)),$$

where

$$\begin{aligned} x(s) &= s, \\ y(s) &= \frac{a^3 s}{b\,(b^2 - a^2)} \left(b\cosh\left(\frac{b}{a}\ln(as)\right) - a\sinh\left(\frac{b}{a}\ln(as)\right)\right), \\ z(s) &= \frac{a^3 s}{b\,(b^2 - a^2)} \left(b\sinh\left(\frac{b}{a}\ln(as)\right) - a\cosh\left(\frac{b}{a}\ln(as)\right)\right); \\ a, b &\in \mathbb{R} - \{0\}. \end{aligned}$$

Here, the equiform differntial vectors are respectively, as follows

$$\begin{aligned} \mathbf{T} &= \left(\frac{s}{a}, \frac{as}{b}\sinh\left(\frac{b}{a}\ln(as)\right), \frac{as}{b}\cosh\left(\frac{b}{a}\ln(as)\right)\right), \\ \mathbf{N} &= \left(0, \frac{s}{a}\cosh\left(\frac{b}{a}\ln(as)\right), \frac{s}{a}\sinh\left(\frac{b}{a}\ln(as)\right)\right), \\ \mathbf{B} &= \left(0, -\frac{s}{a}\sinh\left(\frac{b}{a}\ln(as)\right), -\frac{s}{a}\cosh\left(\frac{b}{a}\ln(as)\right)\right). \end{aligned}$$

Equiform curvature and equiform torsion are calculated as follows

$$\mathcal{K} = \frac{1}{a}, \mathcal{T} = \frac{b}{a^2}.$$

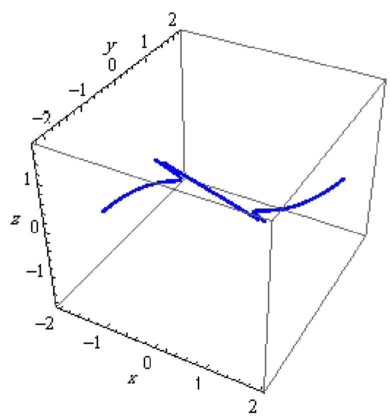

Figure 4: Equiform spacelike circular helix with $\mathcal{K} = \frac{1}{a}, \mathcal{T} = \frac{b}{a^2}$.

Example 6.5 *If we consider the equiform* ***timelike*** *isotropic logarithmic spiral* $\mathbf{r} : I \longrightarrow G_3^1, I \subseteq \mathbb{R}$ *parameterized by the arc length* s *with differential form* $ds = dx$*, given by*

$$\mathbf{r}(x) = (x, y(x), 0),$$

where

$$\begin{aligned} x(s) &= s, \\ y(s) &= \frac{as+b}{a^2}\left(\ln(as+b)-1\right), \\ z(s) &= 0; \\ a,b &\in \mathbb{R}-\{0\}. \end{aligned}$$

For this curve, we get

$$\begin{aligned} \mathbf{r}' &= \left(1, \frac{\ln(as+b)}{a}, 0\right), \\ \mathbf{r}'' &= \left(0, \frac{1}{as+b}, 0\right), \\ \mathbf{r}''' &= \left(0, \frac{-a}{(as+b)^2}, 0\right), \end{aligned}$$

and

$$\begin{aligned} \mathbf{e}_1 &= \left(1, \frac{\ln(as+b)}{a}, 0\right), \\ \mathbf{e}_2 &= (0,1,0), \\ \mathbf{e}_3 &= (0,0,1);\ \kappa = \frac{1}{as+b},\ \tau = 0. \end{aligned}$$

In this case, equiform Frenet vectors and equiform curvatures are as follows

$$\begin{aligned} \mathbf{T} &= \left(as+b, \frac{(as+b)\ln(as+b)}{a}, 0\right), \\ \mathbf{N} &= (0, as+b, 0), \\ \mathbf{B} &= (0,0,as+b),\ \mathcal{K} = a, \mathcal{T} = 0. \end{aligned}$$

respectively.

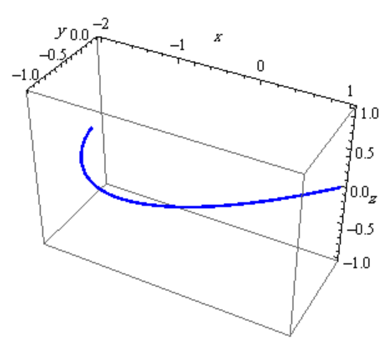

Figure 5: Equiform timelike isotropic logarithmic spiral with $\mathcal{K}(s) = 1, \mathcal{T}(s) = 0$.

From aforementioned calculations, according to (**Proposition 4.2 and Theorems 4.1 − 4.3**), examples $1-4$ are not characterize curves of equiform AW(k), weak equiform AW(2) and weak equiform AW(3)-types. On the other hand, the last example shows that the curve is of equiform AW(2) and AW(3)-types and it is not of equiform AW(1)-type. Also, it is of weak equiform AW(2) and not of weak equiform AW(3)-types.

7 Conclusion

In this paper, we have considered some special curves of equiform AW(k)-type of the pseudo-Galilean 3-space. Also, using the equiform curvature conditions of these curves, the necessary and sufficient conditions for them to be equiform AW(k) and weak equiform AW(k)-types are given. Furthermore, several examples to confirm our main results have been given and illustrated.

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